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A Contour Method on Cayley Trees

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Abstract We consider finite-range lattice models on Cayley trees with two basic properties: the existence of only a finite number of ground states and with a Peierls type condition. We define the notion of a contour for the model on the Cayley tree. By a contour argument we show the existence of s different (where s is the number of ground states) Gibbs measures.

Keywords Cayley tree · Configuration · Ground state · Contour · Gibbs measure

1 Introduction

This paper is a continuation of our previous papers [24–26] devoted to the introduction of a contour method on Cayley trees (Bethe lattices [1]). Lattice systems are widely studied in statistical mechanics; some of them are physically quite realistic, others, such as models on trees, serve as simplifications.

One of the key problems related to lattice spin systems is the description of the set of Gibbs measures. The structure of the lattice plays an important role in the investigations of spin systems. For example in order to study the phase transition problem (non-uniqueness of Gibbs measure) for a system on Z^d and on Cayley trees, respectively, there are two main methods: the contour method (Pirogov–Sinai theory) on Z^d (see e.g. [2, 5, 7, 17, 21–23, 30, 32, 33]) and Markov random field theory on Cayley trees (see e.g. [3, 4, 8–11, 15, 27–29, 31]).

In the Pirogov–Sinai theory configurations can be described by contours which satisfy a Peierls condition. This theory provides tools for a very detailed knowledge of the structure of Gibbs measures in a region in the relevant space of parameters (see e.g. [30]). The

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Pirogov–Sinai theory is a low temperature expansion which enables to control the entropic fluctuations from the ground states, its natural setup being the lattice systems. But the theory is not limited to such cases and it has been applied to a great variety of situations, covering various types of phase transitions (see e.g. [6] for details).

Note that Pirogov–Sinai theory is not simply applicable on Cayley trees and not much work has been done to develop contour methods on trees [24–26]. However studying models with interaction radius $r \geq 2$ on Cayley trees to describe Gibbs measures by the (above-mentioned) method of Markov random field theory becomes difficult, since in this case there appears a set of nonlinear equations which can not solved analytically. To avoid this problem it looks very useful to develop a contour method (Pirogov–Sinai theory) on Cayley trees.

This paper presents a contour method for a general model with a finite interaction radius r ($1 \leq r < \infty$) and with a finite number of ground states with minimal energy density on the Cayley tree of order $k \geq 2$. For $k = 1$ this method was developed in [25] for a model with nonhomogeneous nearest-neighbor interactions.

The paper is organized as follows. In Sect. 2 we give all necessary definitions (Cayley tree, model, Gibbs measure etc). In Sect. 3 under some assumptions on the model (Assumptions A1–A3) we prove the Peierls condition. Section 4 is devoted to definition and properties of contours on Cayley trees. In Sect. 5 by a contour argument we show the existence of s different (where s is the number of ground states) Gibbs measures for the model under consideration on the Cayley tree of order $k \geq 2$. In Sect. 6 we check our assumptions A1–A3 for several examples of models. In the last section we give a discussion about the difference between what the results are on trees, as opposed to amenable lattices.

2 Definitions

2.1 The Cayley Tree

The Cayley tree Γ^k (see [1]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k + 1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighboring vertices*, and we write $l = \langle x, y \rangle$.

The distance $d(x, y)$, $x, y \in V$ on the Cayley tree is defined by the formula

$$d(x, y) = \min\{d \mid \exists x = x_0, x_1, \ldots, x_{d-1}, x_d = y \in V \text{ such that } \langle x_0, x_1 \rangle, \ldots, \langle x_{d-1}, x_d \rangle\}.$$

For a fixed $x^0 \in V$ we set $W_n = \{x \in V \mid d(x, x^0) = n\}$,

$$V_n = \{x \in V \mid d(x, x^0) \leq n\}, \qquad L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}.$$

Denote $|x| = d(x, x^0)$, $x \in V$.

It is known (see e.g. [9]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k + 1$ cyclic groups $\{e, a_i\}$, $i = 1, \ldots, k + 1$ of the second order (i.e. $a_i^2 = e$, $a_i^{-1} = a_i$) with generators $a_1, a_2, \ldots, a_{k+1}$.

Let us define a graph structure on G_k as follows. Vertices which correspond to the "words" $g, h \in G_k$ are called nearest neighbors if either $g = ha_i$ or $h = ga_j$ for some i or j. The graph thus defined is a Cayley tree of order k.

For $g_0 \in G_k$ a left (resp. right) transformation shift on G_k is defined by

$$F_{g_0}h = g_0 h \text{ (resp. } F_{g_0}h = hg_0) \quad \forall h \in G_k.$$

It is easy to see that the set of all left (resp. right) shifts on G_k is isomorphic to G_k.

2.2 Configuration Space and the Model

For $A \subseteq V$ a spin *configuration* σ_A on A is defined as a function $x \in A \to \sigma_A(x) \in \Phi = \{1, 2, \ldots, q\}$; the set of all configurations coincides with $\Omega_A = \Phi^A$. We denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$. Also we define a *periodic configuration* as a configuration $\sigma \in \Omega$ which is invariant under a subgroup of shifts $G_k^* \subset G_k$ of finite index.

More precisely, a configuration $\sigma \in \Omega$ is called G_k^*-periodic if $\sigma(F_y x) = \sigma(x)$ for any $x \in G_k$ and $y \in G_k^*$.

For a given periodic configuration the index of the subgroup is called the *period of the configuration*. A configuration that is invariant with respect to all shifts is called *translational-invariant*.

The energy of the configuration $\sigma \in \Omega$ is given by the formal Hamiltonian

$$H(\sigma) = \sum_{\substack{A \subset V: \\ \operatorname{diam}(A) \le r}} I(\sigma_A) \tag{2.1}$$

where $r \in N = \{1, 2, \ldots\}$, $\operatorname{diam}(A) = \max_{x,y \in A} d(x, y)$, $I(\sigma_A) : \Omega_A \to R$ is a given translation invariant potential i.e. $I(\sigma_A) = I(\sigma_{F_y A})$ for any $y \in G_k$. Here $\sigma_{F_y A} = \{\sigma(F_y x), x \in A\}$.

Fix $r \in N$ and put $r' = [\frac{r+1}{2}]$, where $[a]$ is the integer part of a. Denote by M_r the set of all balls $b_r(x) = \{y \in V : d(x, y) \le r'\}$ with radius r' i.e.

$$M_r = \{b_r(x) : x \in V\}.$$

For $A \subset V$ with $\operatorname{diam}(A) \le r$ denote

$$n(A) = |\{b \in M_r : A \subset b\}|,$$

where $|A|$ stands for the number of elements of a set A.

The Hamiltonian (2.1) can be written as

$$H(\sigma) = \sum_{b \in M_r} U(\sigma_b), \tag{2.2}$$

where $U(\sigma_b) = \sum_{A \subset b} \frac{I(\sigma_A)}{n(A)}$.

For a finite domain $D \subset V$ with the boundary condition φ_{D^c} given on its complement $D^c = V \setminus D$, the conditional Hamiltonian is

$$H(\sigma_D \mid \varphi_{D^c}) = \sum_{\substack{b \in M_r: \\ b \cap D \neq \emptyset}} U(\sigma_b), \tag{2.3}$$

where

$$\sigma_b(x) = \begin{cases} \sigma(x) & \text{if } x \in b \cap D, \\ \varphi(x) & \text{if } x \in b \cap D^c. \end{cases}$$

2.3 The Ground State

A *ground state* of (2.2) is a configuration φ in Γ^k whose energy cannot be lowered by changing φ in some local region. We assume that (2.2) has a finite number of translation-periodic (i.e. invariant under the action of some subgroup of G_k of finite index) ground states. By a standard trick of partitioning the tree into disjoint sets $Q(x)$ centered at $x \in G_k^*$ (the corresponding subgroup of finite index) and enlarging the spin space from Φ to Φ^Q one can transform the model above into a model with only translation-invariant or non-periodic ground states. Such a transformation was considered in [14] for models on Z^d. Hence, without loss of generality, we assume translation-invariance instead of translation-periodicity and we permute the spins so that the set of ground states of the model be $GS = GS(H) = \{\sigma^{(i)}, i = 1, 2, \ldots, s\}$, $1 \le s \le q$ with $\sigma^{(i)}(x) = i$ for any $x \in V$.

2.4 Gibbs Measure

We consider a standard sigma-algebra $\mathcal{B}$ of subsets of Ω generated by cylinder subsets; all probability measures are considered on $(\Omega, \mathcal{B})$. A probability measure μ is called a *Gibbs measure* (with Hamiltonian H) if it satisfies the DLR equation: $\forall\, n = 1, 2, \ldots$ and $\sigma_n \in \Omega_{V_n}$:

$$\mu(\{\sigma \in \Omega : \sigma \mid_{V_n} = \sigma_n\}) = \int_\Omega \mu(\mathrm{d}\omega) \nu_\varphi^{V_n}(\sigma_n), \tag{2.4}$$

where $\nu_\varphi^{V_n}$ is the conditional probability:

$$\nu_\varphi^{V_n}(\sigma_n) = \frac{1}{Z_{n,\varphi}} \exp(-\beta H(\sigma_n \mid \varphi_{V_n^c})). \tag{2.5}$$

Here $\beta = \frac{1}{T}$, $T > 0$—temperature and $Z_{n,\varphi}$ stands for the partition function in V_n, with the boundary condition φ:

$$Z_{n,\varphi} = \sum_{\widetilde{\sigma}_n \in \Omega_{V_n}} \exp(-\beta H(\widetilde{\sigma}_n \mid \varphi_{V_n^c})). \tag{2.6}$$

3 The Peierls Condition

Denote by $\mathbf{U}$ the set of all possible values of $U(\sigma_b)$ for any configuration σ_b, $b \in M_r$. Since $r < +\infty$ we have $|\mathbf{U}| < +\infty$. Put $U^{\min} = \min\{U : U \in \mathbf{U}\}$ and

$$\lambda_0 = \min\{\mathbf{U} \setminus \{U \in \mathbf{U} : U = U^{\min}\}\} - U^{\min}. \tag{3.1}$$

The important assumptions of this paper are the following:

Assumption A1 *The set of all ground states is* $GS = \{\sigma^{(i)}, i = 1, 2, \ldots, s\}$, $1 \le s \le q$.

Assumption A2 $\lambda_0 > 0$.

Assumption A3 *Each* $\varphi \in GS$ *satisfies*

$$U(\varphi_b) = U^{\min} \quad \textit{for every } b \in M_r. \tag{3.2}$$

Remark If a configuration σ satisfies (3.2) i.e. $U(\sigma_b) = U^{\min}$ for $\forall b \in M_r$ then it is a ground state. Moreover for Hamiltonians on Z^d it is well known that a configuration is a ground state if and only if the condition (3.2) is satisfied (see e.g. [30]). But such a fact is not true for Hamiltonians on the Cayley tree, since the tree is a non-amenable graph i.e. $\inf\{\frac{|\text{boundary of } W|}{|W|} : W \subset V, 0 < |W| < \infty\} > 0$ for $k \geq 2$ (see e.g. [1, 12]).

The *relative Hamiltonian* is defined by

$$H(\sigma, \varphi) = \sum_{b \in M_r} (U(\sigma_b) - U(\varphi_b)).$$

Definition 3.1 Let GS be the complete set of all ground states of the relative Hamiltonian H. A ball $b \in M_r$ is said to be an *improper* ball of the configuration σ if $\sigma_b \neq \varphi_b$ for any $\varphi \in GS$. The union of the improper balls of a configuration σ is called the *boundary* of the configuration and denoted by $\partial(\sigma)$.

Definition 3.2 The relative Hamiltonian H with the set of ground states GS satisfies the Peierls condition if for any $\varphi \in GS$ and any configuration σ coinciding almost everywhere with φ (i.e. $|\{x \in V : \sigma(x) \neq \varphi(x)\}| < \infty$)

$$H(\sigma, \varphi) \geq \lambda |\partial(\sigma)|,$$

where λ is a positive constant which does not depend on σ, and $|\partial(\sigma)|$ is the number of balls in $\partial(\sigma)$.

Theorem 3.3 *If assumptions* A1–A3 *are satisfied then the Peierls condition holds.*

Proof Suppose σ coincides almost everywhere with a ground state $\varphi \in GS$ then we have $U(\sigma_b) - U^{\min} \geq \lambda_0$ for any $b \in \partial(\sigma)$ since φ is a ground state. Thus

$$H(\sigma, \varphi) = \sum_{b \in M_r} (U(\sigma_b) - U(\varphi_b)) = \sum_{b \in \partial(\sigma)} (U(\sigma_b) - U^{\min}) \geq \lambda_0 |\partial(\sigma)|.$$

Therefore, the Peierls condition is satisfied for $\lambda = \lambda_0$. The theorem is proved. □

4 Contours on Cayley Tree

Let $\Lambda \subset V$ be a finite set. Let $\sigma^{(i)}_{\Lambda^c} \equiv i$, $i = 1, \ldots, s$ be a constant configuration outside of Λ. For each i we extend the configuration σ_Λ inside Λ to the entire tree by the ith constant configuration and denote it by $\sigma^{(i)}_\Lambda$. The set of such configurations we denote by $\Omega^{(i)}_\Lambda$.

Now we are going to recall a construction of the subcontours (see [24]). Note that our definition (see Definition 4.3 below) of a contour depends on r, at $r = 1$ we get a contour defined in [24]. But the definition of a subcontour does not depend on r.

Consider V_n and for a given configuration $\sigma^{(i)}_\Lambda \in \Omega^{(i)}_\Lambda$ denote $V^{(j)}_n \equiv V^{(j)}_n(\sigma^{(i)}_\Lambda) = \{t \in V_n : \sigma^{(i)}_\Lambda(t) = j\}$, $j = 1, \ldots, q$, $j \neq i$. Let $G^{n,j} = (V^{(j)}_n, L^{(j)}_n)$ be a graph such that

$$L^{(j)}_n = \{l = \langle x, y \rangle \in L : x, y \in V^{(j)}_n\}, \quad j = 1, \ldots, q, j \neq i.$$

It is clear, that for a fixed n the graph $G^{n,j}$ contains a finite number $(= m)$ of maximal connected subgraphs $G^{n,j}_p$ i.e.

$$G^{n,j} = \{G^{n,j}_1, \ldots, G^{n,j}_m\}, \qquad G^{n,j}_p = (V^{(j)}_{n,p}, L^{(j)}_{n,p}), \quad p = 1, \ldots, m; \ j \neq i.$$

Here $V_{n,p}^{(j)}$ and $L_{n,p}^{(j)}$ are the set of vertexes and edges of $G_p^{n,j}$, respectively.

Two edges $l_1, l_2 \in L$ $(l_1 \neq l_2)$ are called *nearest neighboring edges* if $|i(l_1) \cap i(l_2)| = 1$, and we write $\langle l_1, l_2 \rangle_1$.

For any subgraph $K \subset \Gamma^k$ denote by $E(K)$ the set of edges, by $V(K)$ the set of vertices of K and

$$B(K) = \{l \in L \setminus E(K) : \exists l_1 \in E(K) \text{ such that } \langle l, l_1 \rangle_1\}.$$

Definition 4.1 An edge $l = \langle x, y \rangle \in L_{n+1}$ is called a *boundary edge* of the configuration $\sigma_{V_n}^{(i)}$ if $\sigma_{V_n}^{(i)}(x) \neq \sigma_{V_n}^{(i)}(y)$.

The set of boundary edges of $\sigma_{V_n}^{(i)}$ is called *edge boundary* $\partial_1(\sigma_{V_n}^{(i)}) \equiv \partial_1$ of the configuration.

The (finite) set $B(G_p^{n,j})$, $j = 1, \ldots, q$, $j \neq i$, $p = 1, \ldots, m$ (together with a given configuration on it) is called *subcontour* of the boundary ∂_1.

The set $V_{n,p}^{(j)}$, $j = 1, \ldots, q$, $j \neq i$, $p = 1, \ldots, m$ is called *interior* of $B(G_p^{n,j})$, and is denoted by $\operatorname{Int} B(G_p^{n,j})$. The set of edges from a subcontour T is denoted by $\operatorname{supp} T$. The configuration $\sigma_{V_n}^{(i)}$ takes the same value j at all points of the connected component $G_p^{n,j}$. This value $v = v(G_p^{n,j})$ is called the *mark* of the subcontour and denoted by $v(T)$, where $T = B(G_p^{n,j})$.

The collection of subcontours $\tau = \tau(\sigma_{V_n}^{(i)}) = \{T_p\}$ generated by the edge boundary $\partial_1 = \partial_1(\sigma_{V_n}^{(i)})$ of $\sigma_{V_n}^{(i)}$ has the following properties

(a) Every subcontour $T \in \tau$ lies inside the set V_{n+1}.
(b) For every two subcontours $T_1, T_2 \in \tau$ their supports $\operatorname{supp} T_1$ and $\operatorname{supp} T_2$ satisfy $|\operatorname{supp} T_1 \cap \operatorname{supp} T_2| \in \{0, 1\}$.
(c) For any two subcontours $T_1, T_2 \in \tau$ with $|\operatorname{supp} T_1 \cap \operatorname{supp} T_2| = 1$ we have $v(T_1) \neq v(T_2)$.

The distance $\operatorname{dist}(T_1, T_2)$ is defined by

$$\operatorname{dist}(T_1, T_2) = \min_{\substack{x \in V(T_1) \\ y \in V(T_2)}} d(x, y),$$

where $d(x, y)$ is the distance between $x, y \in V$ (see Sect. 2.1).

Recall $r' = [\frac{r+1}{2}]$.

Definition 4.2 The subcontours T_1, T_2 are called *adjacent* if $\operatorname{dist}(T_1, T_2) \leq 2(r' - 1)$. A set of subcontours $\mathcal{A}$ is called *connected* if for any two subcontours $T_1, T_2 \in \mathcal{A}$ there is a collection of subcontours $T_1 = \tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_l = T_2$ in $\mathcal{A}$ such that for each $i = 1, \ldots, l-1$ the subcontours $\tilde{T}_i$ and $\tilde{T}_{i+1}$ are adjacent.

Definition 4.3 Any maximal connected set (component) of subcontours (with given marks) is called a *contour* of the set ∂_1.

For contour $\gamma = \{T_p\}$ denote $\operatorname{Int} \gamma = \bigcup_p \operatorname{Int} T_p$.

Remarks

1. Note that Definition 4.3 of contours coincides with the Definition 2 of [24] for $r = 1$. But Definition 4.3 is better than the corresponding Definition 11 of [26] for $r = 2$. Because

for $r = 2$ from Definition 4.2 we have $\mathrm{dist}(T_1, T_2) = 0$ i.e. the subcontours do not interact if the distance between them is ≥ 1 but in [26] the condition was like $\mathrm{dist}(T_1, T_2) \leq 2$.

2. Our definition of a contour is slightly different from the definition of contour of Hamiltonians on Z^d, $d \geq 2$ (see [22, 23, 30]). For any two contours γ, γ' we have $\mathrm{dist}(\gamma, \gamma') > 2(r' - 1)$. Thus our contours do not interact. This means that for any $\sigma \in \Omega$ there is no a ball $b \in \partial(\sigma)$ with $b \cap \gamma \neq \emptyset$ and $b \cap \gamma' \neq \emptyset$. Such property allows as to use a *contour-removal* operation. This operation is similar to the one in the ordinary Peierls argument [7]: Given a family of contours defining a configuration $\sigma \in \Omega_\Lambda^{(i)}$, the family obtained by omitting one of them is also the family of contours of a (different) configuration in $\Omega_\Lambda^{(i)}$. There is an algorithm of the contour-removal operation to obtain a new configuration as follows. Take the configuration σ and change all the spins in the interior of γ (which must be removed) to value i. This makes γ disappear, but leaves intact the other contours.

For a given (sub)contour γ denote

$$\mathrm{imp}\,\gamma = \{b \in \partial : b \cap \gamma \neq \emptyset\}, \quad |\gamma| = |\mathrm{imp}\,\gamma|.$$

By construction we have $\mathrm{imp}\,\gamma \cap \mathrm{imp}\,\gamma' = \emptyset$ for any contours $\gamma \neq \gamma'$.
For $A \subset V$ denote

$$C(A) = \{b \in M_r : b \cap A \neq \emptyset\},$$

$$D(A) = \{x \in V \setminus A : \exists y \in A, \text{ such that } \langle x, y\rangle\}.$$

Lemma 4.4 *Let K be a connected subgraph of the Cayley tree Γ^k of order $k \geq 2$, such that $|V(K)| = n$, then*

(i) $|D(V(K))| = (k-1)n + 2$.
(ii) $|C(V(K))| = k^{r'-1}((k-1)n + 2)$.

Proof (i) We shall use induction over n. For $n = 1$ and 2 the assertion is trivial. Assume for $n = m$ the lemma is true i.e. from $|V(K)| = m$ follows that $|D(V(K))| = (k-1)m + 2$. We shall prove the assertion for $n = m + 1$ i.e. for $\tilde{K} = K \cup \{x\}$. Since $\tilde{K}$ is connected graph we have $x \in D(V(K))$ and there is a unique $y \in S_1(x) = \{u \in V : d(x, u) = 1\}$ such that $y \in V(K)$. Thus $D(V(\tilde{K})) = (D(V(K)) \setminus \{x\}) \cup (S_1(x) \setminus \{y\})$. Consequently,

$$|D(V(\tilde{K}))| = |D(V(K))| - 1 + k = (k-1)(m+1) + 2.$$

(ii) Using (i) we obtain $|C(V(K))| = u_{r'}$, where $u_{r'}$ is the last term of the collection $u_1, u_2, \ldots, u_{r'}$ which is defined by the following recurrent relations

$$u_l = 2 + (k-1)\sum_{i=0}^{l-1} u_i, \quad l = 1, 2, \ldots, r', \qquad u_0 = n. \tag{4.1}$$

Iterating (4.1) we get $u_1 = (k-1)n + 2$, $u_2 = k((k-1)n + 2)$, then using the induction over l we obtain $u_l = k^{l-1}((k-1)n + 2)$. This completes the proof. □

Let us define a graph structure on M_r as follows. Two balls $b, b' \in M_r$ are connected by an edge if their centers are nearest neighbors. Denote this graph by $G(M_r)$. Note that the graph $G(M_r)$ is a Cayley tree of order $k \geq 1$. Here the vertices of this graph are balls of M_r. Thus Lemma 1.2 of [5] can be reformulated as follows

Lemma 4.5 *Let $\tilde{N}_{n,G}(x)$ be the number of connected subgraphs $G' \subset G(M_r)$ with $x \in V(G')$ and $|V(G')| = n$. Then*

$$\tilde{N}_{n,G}(x) \le (ek)^n.$$

For $x \in V$ we will write $x \in \gamma$ if $x \in V(\gamma)$.
Denote $N_l(x) = |\{\gamma : x \in \gamma, |\gamma| = l\}|$, where as before $|\gamma| = |\mathrm{imp}\,\gamma|$.

Lemma 4.6 *If $k \ge 2$ then*

$$N_l(x) \le C_0\theta^l, \tag{4.2}$$

where $C_0 = 1 + \frac{k+1}{k-1}(k^{r'}-1)$, $\theta = \theta(k,r) = (2ek)^{2(k+1)(r'-1)k^{r'-1}+2}$.

Proof Denote by K_γ the minimal connected subgraph of Γ^k, which contains a contour $\gamma = \{\gamma_1, \dots, \gamma_m\}, m \ge 1$, where γ_i is subcontour. Put

$$\begin{aligned} M^-_{r,\gamma} &= \{x \in \mathrm{Int}\,\gamma : \mathrm{dist}(x, V \setminus \mathrm{Int}\,\gamma) > r'\},\\ M^0_{r,\gamma} &= \{x \in \mathrm{Int}\,\gamma : \mathrm{dist}(x, V \setminus \mathrm{Int}\,\gamma) \le r'\},\\ M^+_{r,\gamma} &= \{x \in V \setminus \mathrm{Int}\,\gamma : \mathrm{dist}(x, \mathrm{Int}\,\gamma) \le r'\},\\ Y_\gamma &= V(K_\gamma) \setminus (\mathrm{Int}\,\gamma \cup D(\mathrm{Int}\,\gamma)). \end{aligned}$$

We have

$$\begin{aligned} |\gamma| &= |M^0_{r,\gamma}| + |M^+_{r,\gamma}|,\\ |C(V(K_\gamma))| &\le |M^-_{r,\gamma}| + |\gamma| + |C(Y_\gamma)|. \end{aligned} \tag{4.3}$$

For any $k \ge 2$, $r \ge 1$ by Lemma 4.4 we have

$$|M^-_{r,\gamma}| = \frac{|D(M^-_{r,\gamma})| - 2}{k-1} < |D(M^-_{r,\gamma})| \le |M^0_{r,\gamma}| < |\gamma|. \tag{4.4}$$

Note that $0 \le |Y_\gamma| \le 2(m-1)(r'-1)$. Thus

$$|C(Y_\gamma)| \le 2(m-1)(r'-1)|C(\{y\})|, \tag{4.5}$$

where y is an arbitrary point of Y_γ. By Lemma 4.4 we have $|C(\{y\})| = k^{r'-1}(k+1)$ since $|V(\{y\})| = 1$. Hence from (4.3–4.5) we get

$$|C(V(K_\gamma))| < 2|\gamma| + 2(m-1)(r'-1)(k+1)k^{r'-1}. \tag{4.6}$$

Since γ contains m subcontours we have $m < |\gamma|$. A combinatorial calculations show that

$$N_l(x) \le C_0 \sum_{m=1}^{l} \binom{2l + 2k^{r'-1}(k+1)(r'-1)(m-1)}{l} \tilde{N}_{2l+2k^{r'-1}(k+1)(r'-1)(m-1),\Gamma^k}(b_x), \tag{4.7}$$

where $\tilde{N}_{l,\Gamma^k}$ is defined in Lemma 4.5 and b_x is a ball $b \in M_r$ such that $x \in b$. Using inequality $\binom{n}{l} \le 2^{n-1}$, $l \le n$ and Lemma 4.5 from (4.7) we get (4.2). The lemma is proved. □

5 Non-uniqueness of Gibbs Measure

For $\sigma_n \in \Omega_{V_n}^{(i)}$ the conditional Hamiltonian (2.3) has the form

$$H^{(i)}(\sigma_n) \equiv H(\sigma_n \mid \sigma_{V_n^c} = i) = \sum_{\substack{b \in M_r: \\ b \cap V_n \neq \emptyset}} U(\sigma_{n,b})$$

$$= \sum_{b \in \partial(\sigma_n)} (U(\sigma_{n,b}) - U^{\min}) + |C(V_n)|U^{\min}, \tag{5.1}$$

where $\sigma_{n,b} = (\sigma_n)_b$.

The Gibbs measure on the space $\Omega_{V_n}^{(i)}$ with boundary condition $\sigma^{(i)}$ is defined as

$$\mu_{n,\beta}^{(i)}(\sigma_n) = \mathbf{Z}_{n,i}^{-1} \exp(-\beta H^{(i)}(\sigma_n)), \tag{5.2}$$

where $\mathbf{Z}_{n,i}$ is the normalizing factor.

Let us consider a sequence of balls on Γ^k

$$V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots, \quad \bigcup V_n = V,$$

and s sequences of boundary conditions outside these balls:

$$\sigma_n^{(i)} \equiv i, \quad n = 1, 2, \ldots, \ i = 1, \ldots, s.$$

By very similar argument as in the proof of Lemma 9.2 in [17] one can prove that each of the s sequences of measures $\{\mu_{n,\beta}^{(i)}, n = 1, 2, \ldots\}$, $i = 1, \ldots, s$ contains a convergent subsequence.

We denote the corresponding limits by $\mu_\beta^{(i)}, i = 1, \ldots, s$. Our purpose is to show for a sufficiently large β these measures are different.

Lemma 5.1 *Suppose assumptions A1–A3 are satisfied. Let γ be a fixed contour and $p_i(\gamma) = \mu_\beta^{(i)}(\sigma_n : \gamma \in \partial(\sigma_n))$. Then*

$$p_i(\gamma) \leq \exp\{-\beta\lambda_0|\gamma|\}, \tag{5.3}$$

where λ_0 is defined by formula (3.1).

Proof Put $\Omega_\gamma = \{\sigma_n \in \Omega_{V_n}^{(i)} : \gamma \subset \partial(\sigma_n)\}$, $\Omega_\gamma^0 = \{\sigma_n : \gamma \cap \partial = \emptyset\}$ and define a (contour-removal) map $\chi_\gamma : \Omega_\gamma \to \Omega_\gamma^0$ by

$$\chi_\gamma(\sigma_n)(x) = \begin{cases} i & \text{if } x \in \operatorname{Int}\gamma, \\ \sigma_n(x) & \text{if } x \notin \operatorname{Int}\gamma. \end{cases}$$

When γ is fixed then the configuration on $\operatorname{Int}\gamma$ also fixed. Therefore the map χ_γ is one-to-one map. For any $\sigma_n \in \Omega_{V_n}^{(i)}$ we have

$$|\partial(\sigma_n)| = |\partial(\chi_\gamma(\sigma_n))| + |\gamma|.$$

Consequently, using (5.1) one finds

$$p_i(\gamma) = \frac{\sum_{\sigma_n\in\Omega_\gamma}\exp\{-\beta\sum_{b\in\partial(\sigma_n)}(U(\sigma_{n,b})-U^{\min})\}}{\sum_{\tilde\sigma_n}\exp\{-\beta\sum_{b\in\partial(\tilde\sigma_n)}(U(\tilde\sigma_{n,b})-U^{\min})\}}$$
$$\le \frac{\sum_{\sigma_n\in\Omega_\gamma}\exp\{-\beta\sum_{b\in\partial(\sigma_n)}(U(\sigma_{n,b})-U^{\min})\}}{\sum_{\tilde\sigma_n\in\Omega^0_\gamma}\exp\{-\beta\sum_{b\in\partial(\tilde\sigma_n)}(U(\tilde\sigma_{n,b})-U^{\min})\}}$$
$$= \frac{\sum_{\sigma_n\in\Omega_\gamma}\exp\{-\beta\sum_{b\in\partial(\sigma_n)}(U(\sigma_{n,b})-U^{\min})\}}{\sum_{\tilde\sigma_n\in\Omega_\gamma}\exp\{-\beta\sum_{b\in\partial(\chi_\gamma(\tilde\sigma_n))}(U(\chi_\gamma(\tilde\sigma_{n,b}))-U^{\min})\}}. \tag{5.4}$$

Since $\sigma_{n,b}=\chi_\gamma(\sigma_{n,b})$, for any $b\in\partial(\sigma_n)\setminus \operatorname{imp}\gamma$ we have

$$\sum_{b\in\partial(\sigma_n)}(U(\sigma_{n,b})-U^{\min}) = S_1+S_2, \tag{5.5}$$

where $S_1=\sum_{b\in\partial(\chi_\gamma(\sigma_n))}(U(\sigma_{n,b})-U^{\min})$; $S_2=\sum_{b\in\operatorname{imp}\gamma}(U(\sigma_{n,b})-U^{\min})$.

By our construction γ is a contour of $\partial(\sigma_n)$ iff $\sigma_n(x)=i$ for any $x\in M^+_{r,\gamma}$. Consequently, $\operatorname{imp}\gamma$ does not depend on $\sigma_n\in\Omega_\gamma$. By assumptions A1–A3 we have $U(\sigma_{n,b})-U^{\min}\ge\lambda_0>0$, for any $b\in\operatorname{imp}\gamma$.

Hence

$$S_2=\sum_{b\in\operatorname{imp}\gamma}(U(\sigma_{n,b})-U^{\min})\ge\lambda_0|\gamma|,\quad \text{for any } \sigma_n\in\Omega_\gamma. \tag{5.6}$$

Thus from (5.4–5.6) one gets (5.3). The lemma is proved. □

Now using Lemmas 4.6 and 5.1 by a very similar argument as in [24] one can prove the following

Lemma 5.2 *If assumptions A1–A3 are satisfied then for fixed $x\in\Lambda$ uniformly in Λ the following relation holds*

$$\mu^{(i)}_\beta(\sigma_\Lambda:\sigma(x)=j)\to 0,\quad j\ne i\quad \text{as } \beta\to\infty.$$

This lemma implies the main result, i.e.

Theorem 5.3 *If A1–A3 are satisfied then for all sufficiently large β there are at least s (= number of ground states) Gibbs measures for the model (2.2) on Cayley tree of order $k\ge 2$.*

6 Examples

In this section we shall give several examples with the properties A1–A3.

6.1 q-Component Models

Note that under some suitable conditions on the parameters of q-component models (with nearest neighbor interactions) on Cayley trees (see [24]) the assumptions A1–A3 are satisfied. In particular, the *ferromagnetic* Ising, Potts and SOS (solid on solid, see [29] for details) models have the properties A1–A3.

Note that the Ising model in a small positive field is an example where the minus state is a ground state with non-minimal energy density. This is a model which violates assumption A3 (when the external field is positive).

6.2 The Potts Model with Competing Interactions ($k = 2, q = 3$)

Consider the Hamiltonian

$$H(\sigma) = J_1 \sum_{\substack{\langle x,y\rangle,\\ x,y\in V}} \delta_{\sigma(x)\sigma(y)} + J_2 \sum_{\substack{x,y\in V:\\ d(x,y)=2}} \delta_{\sigma(x)\sigma(y)}, \tag{6.1}$$

where $J = (J_1, J_2) \in R^2$, $\sigma(x) \in \Phi = \{1, 2, \ldots, q\}$ and δ is the Kronecker's symbol i.e.

$$\delta_{uv} = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

Note that the Ising model with competing interactions (see [26]) is a particular case of the model (6.1). For the model (6.1) with $k = 2$, $q = 3$ we put

$$U(\sigma_b) \equiv U(\sigma_b, J) = \frac{1}{2} J_1 \sum_{\substack{\langle x,y\rangle,\\ x,y\in b}} \delta_{\sigma(x)\sigma(y)} + J_2 \sum_{\substack{x,y\in b:\\ d(x,y)=2}} \delta_{\sigma(x)\sigma(y)}. \tag{6.2}$$

Simple calculations show that

$$\mathbf{U} = \{U(\sigma_b)\} = \left\{ \frac{3}{2}J_1 + 3J_2,\ J_1 + J_2,\ 3J_2,\ \frac{1}{2}J_1,\ J_2,\ \frac{1}{2}J_1 + J_2 \right\}.$$

By similar argument of [26] (pp. 221–223) one can show that for the model (6.1) the assumptions A1–A3 are satisfied if $J \in \{J \in R^2 : J_1 < 0, J_1 + 4J_2 < 0\}$.

6.3 A Model with Interaction Radius $r \geq 1$

For $A \subset V$ let us define a generalized Kronecker symbol as the function $U_0(\sigma_A) : \Omega_A \to \{|A| - 1, |A| - 2, \ldots, |A| - \min\{|A|, |\Phi|\}\}$ by

$$U_0(\sigma_A) = |A| - |\sigma_A \cap \Phi|, \tag{6.3}$$

where as before $\Phi = \{1, 2, \ldots, q\}$ and $|\sigma_A \cap \Phi|$ is the number of different values of $\sigma_A(x)$, $x \in A$. For instance if σ_A is a constant configuration then $|\sigma_A \cap \Phi| = 1$.

Note that if $|A| = 2$, say, $A = \{x, y\}$, then $U_0(\{\sigma(x), \sigma(y)\}) = \delta_{\sigma(x)\sigma(y)}$.

Now consider the Hamiltonian

$$H(\sigma) = -J \sum_{b\in M_r} U_0(\sigma_b), \tag{6.4}$$

where $J \in R$.

It is easy to see that if $J > 0$ then the assumptions A1–A3 are satisfied for any $r \geq 1$ and $k \geq 2$.

Thus (by Theorem 5.3) these models have q different Gibbs measures for all sufficiently large β.

7 Discussion

Describing the set of all Gibbs distributions corresponding to a given Hamiltonian is one of the main problems of statistical mechanics, but it has not yet been completely solved, even for some rather simple Hamiltonians (see, e.g., [3, 8, 9]). The simplicity of the Cayley tree allows describing a rather wide class of Gibbs measures for models without "good" symmetries (see [15] and [28]) and with competing interactions (see, e.g., [19, 20]).

Note that even if all ground states have the same energy density, this does not imply that all Gibbs measures (mentioned in Theorem 5.3) have the same free energy density.

Now we are going to give a discussion about the difference between what the results are on trees, as opposed to amenable lattices. First difference is related to the SOS model with $m \geq 2$ spin values on Cayley trees. The results of [27] allow indicating the explicit values of β for which there are many Gibbs measures. On the other hand, in view of the results in [29] and in [27], it can be noted that the number of translation-invariant Gibbs measures (for the SOS model on Cayley trees) is the same and is equal to three for $m = 2$ and $m = 3$. This demonstrates an essential distinction between the SOS models on Z^d and on Γ^k because the number of translation-invariant measures in the case of Z^d depends on the parity of m (see [16]), namely, a unique Gibbs measure for even m and two periodic Gibbs measure for odd m. Note that the SOS model can be treated as a natural generalization of the Ising model (obtained for $m = 2$) and of the Blume–Capel model (obtained for $m = 3$). The Blume-Capel model is an example with different-free-energy Gibbs measures. In particular, the main result (Theorem 5.3) of the present paper (see also [24]) shows that (for the Cayley tree case) the independence on m of the number of translation-invariant Gibbs measures is true for any $m \geq 2$.

Second difference: it is known (cf. [13]) that the Potts model with $q \geq 2$ spin values on $Z^d, d \geq 2$ undergoes a first-order phase transition at a certain transition temperature $T_{\rm cr} = T_{\rm cr}(q)$, provided q is large enough. Namely, the model (on Z^d) has q different Gibbs measures for temperatures $T < T_{\rm cr}$, $q + 1$ measures at $T = T_{\rm cr}$ and one measure for $T > T_{\rm cr}$. Note that [8, 18] for the ferromagnetic Potts model with q spin values on a Cayley tree for any $q \geq 2$ (even for $q = 2$, i.e., for the Ising model [3, 8]) there are $q + 1$ distinct translation-invariant Gibbs measures. Namely, there are two critical temperatures $0 < T_c' < T_c$ such that (i) for $T \in (T_c', T_c]$ there are $q + 1$ extreme Gibbs measures. One of them, say μ_0, (with $\mu_0(\sigma : \sigma(x) = i) = \frac{1}{q}$, $i = 1, \ldots, q$) is called unordered Gibbs measure; (ii) for $T \in (0, T_c']$ the $q + 1$ Gibbs measures still exist but the measure μ_0 is not extreme; (iii) for $T > T_c$ there is one Gibbs measure.

The next difference is given in Remark after (3.2): if a configuration σ satisfies $U(\sigma_b) = U^{\min}$ for $\forall b \in M_r$ then it is a ground state. Moreover for Hamiltonians on Z^d it is well known that a configuration is a ground state if and only if the condition (3.2) is satisfied (see e.g. [30]). But such a fact is not true for Hamiltonians on the Cayley tree, since the tree is a non-amenable graph. Assumption A1 gives some restriction i.e. it reduces consideration of the model on the set of parameters where there are only translation-invariant ground states (see Sect. 2.3). In Sect. 6 for some examples the corresponding set of parameters are given. For example, Hamiltonian (6.1) satisfies Assumption A1 if the parameters J are taken from $J \in \{J \in R^2 : J_1 < 0, J_1 + 4J_2 < 0\}$.

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References

1. Baxter, R.J.: Exactly Solved Models in Statistical Mechanics. Academic Press, London (1982)
2. Biskup, M., Borgs, C., Chayes, J.T., Kotecký, R.: Partition function zeros at first-order phase transitions: Pirogov–Sinai theory. J. Stat. Phys. **116**, 97–155 (2004)
3. Bleher, P.M., Ganikhodjaev, N.N.: On pure phases of the Ising model on the Bethe lattice. Theory Probab. Appl. **35**, 216–227 (1990)
4. Bleher, P.M., Ruiz, J., Schonmann, R.H., Shlosman, S., Zagrebnov, V.A.: Rigidity of the critical phases on a Cayley tree. Mosc. Math. J. **3**, 345–362 (2001)
5. Borgs, C.: Statistical physics expansion methods in combinatorics and computer science. http://research.microsfort.com/~borgs/CBMS.pdf (2004)
6. Bovier, A., Merola, I., Presutti, E., Zahradnik, M.: On the Gibbs phase rule in the Pirogov–Sinai regime. J. Stat. Phys. **114**, 1235–1267 (2004)
7. Fernández, R.: Contour ensembles and the description of Gibbsian probability distributions at low temperature. www.univ-rouen.fr/LMRS/persopage/Fernandez (1998)
8. Ganikhodjaev, N.N.: On pure phases of the ferromagnet Potts with three states on the Bethe lattice of order two. Theor. Math. Phys. **85**, 163–175 (1990)
9. Ganikhodjaev, N.N., Rozikov, U.A.: A description of periodic extremal Gibbs measures of some lattice models on the Cayley tree. Theor. Math. Phys. **111**, 480–486 (1997)
10. Ganikhodjaev, N.N., Rozikov, U.A.: The Potts model with countable set of spin values on a Cayley tree. Lett. Math. Phys. **75**, 99–109 (2006)
11. Georgii, H.-O.: Gibbs Measures and Phase Transitions. de Gruyter, Berlin (1988)
12. Grimmett, G.: The Random-Cluster Model. Springer, Berlin (2006)
13. Kotecky, R., Shlosman, S.B.: First-order phase transition in large entropy lattice models. Commun. Math. Phys. **83**, 493–515 (1982)
14. Lebowitz, J.L., Mazel, A.E.: On the uniqueness of Gibbs states in the Pirogov–Sinai theory. Commun. Math. Phys. **189**, 311–321 (1997)
15. Martin, J.B., Rozikov, U.A., Suhov, Y.M.: A three state hard-core model on a Cayley tree. J. Nonlinear Math. Phys. **12**, 432–448 (2005)
16. Mazel, A.E., Suhov, Y.M.: Random surfaces with two-sided constraints: an application of the theory of dominant ground states. J. Stat. Phys. **64**, 111–134 (1991)
17. Minlos, R.A.: Introduction to Mathematical Statistical Physics. University Lecture Series, vol. 19. AMS, Providence (2000)
18. Mossel, E.: Survey: information flow on trees. In: Graphs, Morphisms and Statistical Physics. DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 63, pp. 155–170. AMS, Providence (2004)
19. Mukhamedov, F.M., Rozikov, U.A.: On Gibbs measures of models with competing ternary and binary interactions and corresponding von Neumann algebras. J. Stat. Phys. **114**, 825–848 (2004)
20. Mukhamedov, F.M., Rozikov, U.A.: On Gibbs measures of models with competing ternary and binary interactions and corresponding von Neumann algebras. J. Stat. Phys. **119**, 427–446 (2005)
21. Peierls, R.: On Ising model of ferro magnetism. Proc. Cambr. Phil. Soc. **32**, 477–481 (1936)
22. Pirogov, S.A., Sinai, Y.G.: Phase diagrams of classical lattice systems. Theor. Math. Phys. **25**, 1185–1192 (1975)
23. Pirogov, S.A., Sinai, Y.G.: Phase diagrams of classical lattice systems. Theor. Math. Phys. **26**, 39–49 (1976)
24. Rozikov, U.A.: On q-component models on Cayley tree: contour method. Lett. Math. Phys. **71**, 27–38 (2005)
25. Rozikov, U.A.: An example of one-dimensional phase transition. Sib. Adv. Math. **16**, 121–125 (2006)
26. Rozikov, U.A.: A constructive description of ground states and Gibbs measures for Ising model with two-step interactions on Cayley tree. J. Stat. Phys. **122**, 217–235 (2006)
27. Rozikov, U.A., Shoyusupov, S.A.: Gibbs measures for the SOS model with four states on a Cayley tree. Theor. Math. Phys. **149**, 1312–1323 (2006)
28. Rozikov, U.A., Suhov, Y.M.: A hard-core model on a Cayley tree: an example of a loss network. Queueing Syst. **46**, 197–212 (2004)
29. Rozikov, U.A., Suhov, Y.M.: Gibbs measures for SOS model on a Cayley tree. Inf. Dim. Anal. Quant. Prob. Rel. Fields. **9**, 471–488 (2006)
30. Sinai, Y.G.: Theory of Phase Transitions: Rigorous Results. Pergamon, Oxford (1982)
31. Zachary, S.: Countable state space Markov random fields and Markov chains on trees. Ann. Probab. **11**, 894–903 (1983)
32. Zahradnik, M.: An alternate version of Pirogov–Sinai theory. Commun. Math. Phys. **93**, 559–581 (1984)
33. Zahradnik, M.: A short course on the Pirogov–Sinai theory. Rendi. Math. Ser. VII **18**, 411–486 (1998)